Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Computational Algebraic Geometry and Switching Surfaces in Optimal Control

Uli Walther
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455

Tryphon Georgiou and Allen Tannenbaum
Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455

Abstract

A number of problems in control can be reduced to finding suitable real solutions of algebraic equations. In particular, such a problem arises in the context of switching surfaces in optimal control. Recently, a powerful new methodology for doing symbolic manipulations with polynomial data has been developed and tested, namely the use of Groebner bases. In this note, we apply the Groebner basis technique to find effective solutions to the classical problem of time-optimal control.

1 Introduction

Optimal control is one of the most widely used and studied methodologies in modern systems theory. As is well-known, time-optimal problems lead to switching surfaces which typically are defined or may be approximated by polynomial equations [4, 6]. The problem of determining on which side a given trajectory is in relation to the switching surface is of course key in developing the control strategy. Since the complexity of the switching surfaces can grow to be quite large, this may become quickly a formidable task. Here is where new techniques in computational algebraic geometry may become vital in effectively solving this problem. Thus while there have been a number of interesting more *ad hoc* approaches to the computation of switching surfaces (see [4, 6] and the references therein), we feel that the techniques presented here can systematize the calculations. In this paper, we would like to introduce Groebner bases in the context of optimal control which will reduce the switching surface problem to a combinatorial one. Groebner bases have already been employed in a number of applications in robotics and motion planning [3].

In addition, to the computations of switching surfaces, this paper is intended to be of a tutorial nature. Our main purpose is to introduce a fundamental technique in computational geometry in order to solve an important problem in systems.

This work was supported in part by grants from the National Science Foundation ECS-99700588, ECS-9505995, NSF-LIS, Air Force Office of Scientific Research AF/F49620-98-1-0168, by the Army Research Office DAAG55-98-1-0169, and MURI Grant.

2 Switching Surfaces in Optimal Control

We focus on the classical problem of time-optimal control for a system consisting of a chain of integrators. It is standard that for such a system, minimum-time optimal control with a bounded input, leads to "bang-bang" control with at most n switchings - n being the order of the system. The control algorithm usually requires explicit determination of the switching surfaces where the sign of the control input changes. Explicit expressions for switching strategy are in all but the simplest cases prohibitively complicated (e.g., see [4], [6]).

Consider the linear system with saturated control input

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= u(t), \text{ where } |u(t)| \le 1,\end{aligned}$$

and as objective to drive the system from an initial condition $x(0)$ to a target $x(t_f)$, in minimum time t_f. In this case the Hamiltonian is

$$\mathcal{H} = 1 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u.$$

The co-state equations become

$$\begin{aligned}\dot{\lambda}_1(t) &= 0\\ \dot{\lambda}_2(t) &= -\lambda_1(t)\\ \dot{\lambda}_3(t) &= -\lambda_2(t),\end{aligned} \tag{1}$$

while the optimal $u(t)$ is given by $u(t) = -\text{sign}\,(\lambda_3(t))$.

A closed form expression for the optimal $u(t)$ as a function of $x(t)$ can be worked out (e.g., [4], see also [6]). Such an expression in fact tests the location of the state vector with regard to a switching surface. Bang-bang switching in practice is not desirable because of the incapacitating effect of noise and chattering. While various remedies have been proposed and applied, the basic issue of knowing the switching surfaces is still instrumental in most methodologies.

The approach we take herein is algebraic in nature. The idea is to test directly whether a particular switching strategy is feasible. There are only two possible strategies where the input alternates between $+1$ and -1, taking the values $+1, -1, +1, \ldots$, or $-1, +1, -1, \ldots$, respectively. In each case, taking into account the maximal number of switchings, one can easily derive an expression for the final value of the state as a function of the switching times. This expression is then analyzed against the requirement of a given $x(t_f)$.

For this standard time-optimal control problem, it is well-known and easy to see by analyzing (1) that, in general, there are no singular intervals, and that the control input switches at most 3 times. Designate by t_1, t_2 and t_3, the length of the successive intervals where $u(t)$ stays constant. Any set of initial and final conditions can be translated to having $x(0) = 0$ and a given value for $x(t_f)$ and this is the setting from here on. The particular choice (among the only two possible ones),

$$u(t) = \begin{cases} +1 \text{ for } 0 \le t < t_1 \\ -1 \text{ for } t_1 \le t < t_1 + t_2 \\ +1 \text{ for } t_1 + t_2 \le t < t_1 + t_2 + t_3 =: t_f \end{cases}$$

drives the chain of integrators for the origin to the final point $x(t_f)$ given by

$$\begin{aligned} x_3(t_f) &= t_1 - t_2 + t_3 \\ x_2(t_f) &= \frac{t_1^2}{2} + t_1 t_2 - \frac{t_2^2}{2} - t_2 t_3 + \frac{t_3^2}{2} + t_3 t_1 \\ x_1(t_f) &= \frac{t_1^3}{6} - \frac{t_2^3}{6} + \frac{t_3^3}{6} \\ &\quad + \frac{t_1^2}{2} t_2 + \frac{t_1^2}{2} t_3 + \frac{t_2^2}{2} t_1 - \frac{t_2^2}{2} t_3 + \frac{t_3^2}{2} t_1 \\ &\quad - \frac{t_3^2}{2} t_2 + t_1 t_2 t_3. \end{aligned} \tag{2}$$

It turns out that the selection between alternating values $+1, -1, +1, \ldots$ or, $-1, +1, -1, \ldots$ for the optimal input $u(t)$ depends on whether (2) have a solution for a specified final condition $x(t_f) = (x_1,\ x_2,\ x_3)'$.

3 Computational Algebraic Geometry and Groebner Bases

Algebraic geometry is concerned with the properties of geometric objects (*varieties*) defined as the common zeros of systems of polynomials. More precisely, let k denote a field (e.g., the fields of complex numbers $\mathbf{C}$, real numbers $\mathbf{R}$, or rational numbers $\mathbf{Q}$). Over an algebraically closed field such as $\mathbf{C}$, one may show that affine geometry (the study of subvarieties of affine space k^n) is equivalent to the ideal theory of the polynomial ring $k[x_1, \ldots, x_n]$ (see [3] especially the discussion of the Hilbert Nullenstellensatz). Clearly, the ability to manipulate polynomials and to understand the geometry of the underlying varieties can be very important in a number of applied fields (e.g., the kinematic map in robotics is typically polynomial). We show how the problem in optimal control discussed above, may be reduced to a problem in affine geometry. We follow the treatments in [1, 3].

3.1 Groebner Bases

Motivated by the long division in the polynomial ring of one variable, one needs to order monomials in polynomial rings of several variables $k[x_1, \ldots, x_n]$.

Let ${\mathbf{Z}_+}^n$ denote the set of n-tuples of non-negative integers. Let $\alpha, \beta \in {\mathbf{Z}_+}^n$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, and set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $>$ denote a total (linear) ordering on ${\mathbf{Z}_+}^n$ (this means that exactly one of the following statements is true: $\alpha > \beta$, $\alpha < \beta$, or $\alpha = \beta$). Moreover we say that $x^\alpha > x^\beta$ if $\alpha > \beta$. Then a *monomial ordering* on ${\mathbf{Z}_+}^n$ is a total ordering such that if $\alpha > \beta$ and $\gamma \in {\mathbf{Z}_+}^n$, then $\alpha + \gamma > \beta + \gamma$, and $>$ is a *well-ordering*, i.e., every nonempty subset of ${\mathbf{Z}_+}^n$ has a smallest element. One of the most commonly used monomial ordering is that defined by the ordinary lexicographical order $>_{lex}$ on ${\mathbf{Z}_+}^n$. Recall that this means $\alpha >_{lex} \beta$ if the left most non-zero element of $\alpha - \beta$ is positive.

We now fix a monomial order on ${\mathbf{Z}_+}^n$.Then the *multidegree* of an element $f = \sum_\alpha a_\alpha x^\alpha \in k[x_1, \ldots, x_n]$ (denoted by multideg(f)) is defined to be the maximum α such that $a_\alpha \neq 0$. The *leading term* of f (denoted by LT(f)) is the monomial $a_{\text{multideg}(f)} \cdot x^{\text{multideg}(f)}$. We now come to the following crucial definition:

Definition. A finite set of polynomials $f_1, \ldots, f_m$ of an ideal $I \subset k[x_1, \ldots, x_n]$ is called a *Groebner basis* if the ideal generated by LT(f_i) for $i = 1, \ldots, m$ is equal to the ideal generated by the leading terms of all the elements of I.

The crucial result is that:

Theorem 1 *Every non-trivial ideal has a Groebner basis. Moreover, any Groebner basis of I is a basis of I.*

Notice that the use of Groebner bases reduces the study of generators of polynomial ideals (and so affine algebraic geometry) to that of the combinatorial properties of monomial ideals. Therein lies the power of this method assuming that one can easily compute a Groebner basis (see [1, 3]).

In what follows, we will indicate how Groebner basis techniques may be used to solve polynomial equations.

3.2 Elimination Theory

Elimination theory is a classical method in algebraic geometry for eliminating variables from systems of polynomial equations and as such is a key method in finding their solutions. Groebner bases give a powerful method for carrying out this procedure systematically. We work over an algebraically closed field k in this section.

More precisely, let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The jth *elimination ideal* of I is defined to be

$$I_j = I \cap k[x_{j+1}, \ldots, x_n].$$

Suppose that I is generated by $f_1, \ldots, f_m$. Then I_j is the set of all consequences of the solutions of $f_1 = \ldots = f_m = 0$ in which the variables $x_1, \ldots, x_j$ are eliminated. Thus in order to eliminate $x_1, \ldots, x_j$, we need to find nonzero polynomials in I_j. This is where the Groebner basis methodology plays the key role:

Theorem 2 (Elimination Theorem) *For $I \subset k[x_1, \ldots, x_n]$ an ideal, and G a Groebner basis with respect the the lexicographical order with $x_1 > \ldots > x_n$, for every $j = 0, \ldots, n$*

$$G_j := G \cap k[x_1, \ldots, x_n]$$

is a Groebner basis of I_j. (Note we take $I_0 = I$.)

Thus using Theorem 2, we may eliminate the variables one at a time until we are left with a polynomial in x_n, which we may solve. We must of course then extend the solution to the original system. For an ideal I we set

$$V(I) := \{(z_1, \ldots, z_n) \in k^n : f(z_1, \ldots, z_n) \, \forall f \in I\}.$$

Again this can be done in a systematic matter via the following result.

Theorem 3 (Extension Theorem) *Let $I \subset k[x_1, \ldots, x_n]$ be generated by $f_1, \ldots, f_m$. Let I_1 be the first elimination ideal of I as defined above. For each $i = 1, \ldots, m$ write f_i as*

$$f_i = g_i(x_2, \ldots, x_n) x_1^{n_1} + \textit{lower order terms in } x_1.$$

Suppose that $(z_2, \ldots, z_n) \in V(I_1)$. Then if there exists some i such that $g_i(z_2, \ldots, z_n) \neq 0$, then we may extend $(z_2, \ldots, z_n)$ to a solution of $(z_1, \ldots, z_n) \in V(I)$.

This ends our brief discussion of Groebner bases and elimination theory. We should note that there are symbolic implementations of this methodology on such standard packages as Mathematica, Maple, or Macaulay [5].

4 Computation of Switching Surfaces

In this section, we indicate the solution to the time optimal control problem formulated in Section 2. Even though we work out the case of 3rd order system, the method we propose is completely general, and should extend in a straightforward manner to any number of switchings.

In what follows below, we set

$$x := t_1, \; y := t_2, \; z := t_3,$$

and

$$a := x_3(t_f), \; b := x_2(t_f), c := x_3(t_f).$$

4.1 Complex Solutions

In this section, we solve the complex version of the switching problem, namely:

Problem 1 *Given is the system of equations*

$$x - y + z = a,$$
$$\frac{x^2}{2} + xy + \frac{z^2}{2} + zx - \frac{y^2}{2} - yz = b,$$
$$\frac{x^3}{6} + \frac{z^3}{6} + \frac{x^2y}{2} + \frac{x^2z}{2} + \frac{y^2x}{2} + \frac{z^2x}{2} + xyz - \frac{y^3}{6} - \frac{y^2z}{2} - \frac{z^2y}{2} = c.$$

We shall first be interested in solving the following question:

- *If $a, b, c \in \mathbf{C}$, does the system have complex solutions x, y, z?*

The answer will be yes.

To illustrate the use of the Macaulay symbolic program in computational algebraic geometry, we will put in some of the relevant scripts. Let us call I the ideal in $\mathbf{Q}[x, y, z, a, b, c]$ generated by the three forms above. As a first step, let us compute a Groebner basis for I. We introduce the elimination order $x >> y >> z >> c >> b >> a$. Here is a Macaulay command sequence to accomplish this:

```
1% ring R
! characteristic (if not 31991)        ?
! number of variables                  ? 6
!   6 variables, please                ? xyzcba
! variable weights (if not all 1)      ?
! monomial order (if not rev. lex.)    ?
1 1 1 1 1 1
largest degree of a monomial          :
512 512 512 512 512 512
```

```
1% <ideal I x-y+z-a x2/2+xy+z2/2+zx-y2/2-yz-b\
 x3/6+z3/6+x2y/2+x2z/2+y2x/2+z2x/2+xyz-y3/6-\\
y2z/2-z2y/2-c
1% <inhomog_std I II
```

The result is the following 7 forms, made visible by

```
putstd II
```

$$
\begin{aligned}
&z^4b - 1/2z^4a^2 - 2z^3c - 2z^3ba + 4/3z^3a^3 + \\
&6z^2ca + z^2b^2 - z^2ba^2 - 3/4z^2a^4 \\
&\quad - 4zcb - 4zca^2 + 2zb^2a + 2/3zba^3 + 1/6za^5 + c^2 + \\
&2cba + 2/3ca^3 - b^3 \\
&\quad - 1/2b^2a^2 - 1/12ba^4 - 1/72a^6, \qquad (3)\\
&yc^2 - 2ycba + 2/3yca^3 + yb^3 - 1/2yb^2a^2 + 1/12yba^4 - \\
&1/72ya^6 + z^3b^2 - z^3ba^2 \\
&\quad + 1/4z^3a^4 - z^2cb + 1/2z^2ca^2 - \\
&2z^2b^2a + 13/6z^2ba^3 - 7/12z^2a^5 \\
&\quad - 2zc^2 + 6zcba - 7/3zca^3 - zb^2a^2 + \\
&7/36za^6 + 2c^2a - 2cb^2 - 3cba^2 \\
&\quad + 4/3ca^4 + 2b^3a - 2/3b^2a^3 - 1/36a^7, \qquad (4)\\
&yzb - 1/2yza^2 - yc + 1/6ya^3 - z^2b + 1/2z^2a^2 + \\
&2zc - 1/3za^3 - 2ca + b^2 + 1/12a^4, \qquad (5)\\
&yzc - 1/6yza^3 - 2yca + yb^2 + 1/12ya^4 + z^3b - \\
&1/2z^3a^2 - 2z^2c - 2z^2ba + 4/3z^2a^3 \\
&\quad + 6zca - zba^2 - 1/2za^4 - cb - 7/2ca^2 + \\
&2b^2a + 1/6ba^3 + 1/12a^5, \qquad (6)\\
&yz^2 - 2yza + yb + 1/2ya^2 + zb - 1/2za^2 - c + 1/6a^3 \qquad (7)\\
&y^2 - 2yz + 2ya - b + 1/2a^2, \qquad (8)\\
&x - y + z - a. \qquad (9)
\end{aligned}
$$

At this point we remark that the Groebner basis would look just the same if we had considered the extension of the ideal to the ring of polynomials over $\mathbf{R}$ or $\mathbf{C}$. This is true in general.

Now if the three forms from Problem 1 have a solution, then certainly the quartic given by (3) above, also must have a solution, whatever the base field. Over $\mathbf{C}$ this will have a solution for sure if the leading form is nonzero, which is the case if and only if $a^2 - 2b \neq 0$.

Moreover, if the quartic (3) does indeed have a solution over $\mathbf{C}$ (i.e. $\exists z \in \mathbf{C}$ that makes the equation true for chosen $a, b, c \in \mathbf{C}$), then the Extension Theorem tells us from the last two equations (8, 9) of the preceding script, that we can then find y and then x in $\mathbf{C}$ solving the entire system over $\mathbf{C}$.

Let us continue to investigate the question whether we can find $z \in \mathbf{C}$ such that the quartic holds true in the case where $a^2 = 2b$. In that case, we need to add $a^2 - 2b$ to the generators of our ideal, and recompute the Groebner basis. Here is the script:

```
1% <ideal J a^2-2b
1% concat J I
1% <inhomog_std J JJ
```

In this case the output is

$$
\begin{aligned}
&b - 1/2a^2, \\
&z^3c - 1/6z^3a^3 - 3z^2ca + 1/2z^2a^4 + 3zca^2 - \\
&1/2za^5 - 1/2c^2 - 5/6ca^3 + 11/72a^6, \\
&yc - 1/6ya^3 - 2zc + 1/3za^3 + 2ca - 1/3a^4, \\
&yz^2 - 2yza + ya^2 - c + 1/6a^3, \\
&y^2 - 2yz + 2ya, \\
&x - y + z - a.
\end{aligned}
$$

Not surprisingly, the quartic became a cubic when we set the leading coefficient to zero. As before, the cubic will have a complex (even real) root as long as the leading coefficient $a^3 - 6c$ is nonzero. And also as before, the two last equations ensure that each solution for z may be extended to (x, y, z) solving the system.

What happens if $a^3 = 6c$? Let us add this relation and recompute a Groebner basis:

```
1% <ideal K a^3-6c
1% concat K J
1% <inhomog_std K KK
```

which leads to $b - 1/2a^2, c - 1/6a^3, yz^2 - 2yza + ya^2, y^2 - 2yz + 2ya, x - y + z - a$. A somewhat surprising thing happened: when we killed the leading coefficient of the cubic, the entire polynomial died. Let us factor as much as we can in the output: $b - 1/2a^2, c - 1/6a^3, y(z - a)^2, y(y - 2(z - a)), x - y + (z - a)$. One can see that this system has for complex or real a, b, c for example the solution $(x, y, z) = (0, 0, a)$.

We conclude:

- The system does always have a complex solution.
- If $a^2 = 2b, a^3 = 6c$ and $a, b, c \in \mathbf{R}$, the system has real solutions.
- If $a^2 = 2b, a^3 = 6c$ and $0 \leq a, b, c \in \mathbf{R}$, the system has real nonnegative solutions.

4.2 Real Positive Solutions

Now that we have established the existence of complex solutions x, y, z for any parameter set (a, b, c) let us search for the existence of real nonnegative solutions for real parameters. This will solve our switching control problem. Thus as a second step we will answer the following:

Problem 2 *Given are $a, b, c \in \mathbf{R}$. Does there exist a nonnegative solution vector (x, y, z) for the system (1) in the sense that $x \geq 0, y \geq 0, z \geq 0$?*

Thus, if there is a positive solution x, y, z, then the value of the optimal control u assumes the values $+1, -1, +1$ successively, and in particular, the present value for the optimal control is $u(0) = +1$. If no positive solution exists the present value of the optimal control is $u(0) = -1$. The techniques we will use are computations of suitable Groebner bases together with an algorithm from real algebraic geometry called *Sturm sequences.* Sturm sequences are associated to polynomials as follows. Suppose $f(x)$ is a single variable polynomial with real coefficients. We define $p_0(x) = f(x)$, $p_1(x) = f'(x)$, and then recursively p_i by $p_i = q_{i-1}p_{i-1} - p_{i-2}$ for $i > 1$. Here we demand that $\deg(p_i) < \deg(p_{i-1})$. So, p_i is up to sign the remainder of Euclidean division of p_{i-2} by p_{i-1}.

Theorem 3 *([2], Theorem 1.2.) Let $\alpha < \beta$ be real numbers which are not roots of $f(x)$. Define a function $v(\gamma)$ for $\gamma \in \mathbf{R}$ by counting the number of sign changes in the sequence $\{p_i(\gamma)\}$, $i \geq 0$, dropping all zeros. Then $v(\alpha) - v(\beta)$ is the number of distinct zeros of f between α and β.*

The significance of the theorem for us lies in the fact that although it does not specify the location of the zeros it gives a qualitative answer, which as pointed out above is all we need to know about for the purpose of dynamical steering.

As a first step we compute a Groebner basis for the three polynomials in (1) under an elimination order $x >> z >> y >> c >> b >> a$. Note the switch of the variables y and z in the ordering. One gets

$$\begin{aligned} &y^4 + 4y^2b - 2y^2a^2 - 4yc + 4yba - \\ &\quad 4/3ya^3 - b^2 + ba^2 - 1/4a^4, &(10)\\ &zb - 1/2za^2 + 1/2y^3 + 3/2yb - \\ &\quad 3/4ya^2 - 2c + ba - 1/6a^3, &(11)\\ &zy - 1/2y^2 - ya + 1/2b - 1/4a^2, &(12)\\ &x + z - y - a. &(13)\end{aligned}$$

This suggests that one ought to solve equation (11) or (12) for z:

$$z = \frac{-1/2y^3 - 3/2yb + 3/4ya^2 + 2c - ba + 1/6a^3}{b - 1/2a^2},$$

$$z = \frac{y^2/2 + ya - 1/2b + 1/4a^2}{y},$$

respectively. This of course is assuming that y and $b - a^2/2$ are not zero. It is easy to check that these solutions for z are not contradicting each other. In fact, they differ by a multiple of the quartic in y, given in (10).

One sees that $y = 0$ is equivalent to $2b - a^2 = 6c - a^3 = 0$. These relations simplify the system to

$$\begin{aligned} &b - 1/2a^2,\\ &c - 1/6a^3,\\ &y^3, &(14)\\ &zy + 1/ - 2y^2 - ya,\\ &x + z - y - a.\end{aligned}$$

This has the solutions $y = 0$, $z =$arbitrary, $x = a - z$. Since $y = 0$ is equivalent to $a^3 - 6c = a^2 - 2b = 0$, testing the latter conditions is sufficient to find out whether $y = 0$. In that case nonnegative solutions will exist precisely when a is nonnegative. This covers the case $y = 0$. If $x = 0$ we have the system

$$\begin{aligned} &c^2 - 2cba + 2/3ca^3 + b^3 - 1/2b^2a^2 + 1/12ba^4 - 1/72a^6,\\ &yb - 1/2ya^2 - c + ba - 1/3a^3,\\ &yc - 1/6ya^3 - ca + b^2 - 1/12a^4, &(15)\\ &y^2 + b - 1/2a^2,\\ &z - y - a,\\ &x.\end{aligned}$$

Since a, b, c are known it is easy to check the consistency of this system, by solving each of the three middle equations for y and testing the vanishing of the first. If consistency fails, we are not in the case $x = 0$. If the sytem is consistent, one needs to check whether the obtained solutions for y, z are nonnegative. If that is so set $u = 1$ and otherwise $u = -1$, finishing the case $x = 0$.

In a similar fashion one does the case $z = 0$. If $z = 0$ one gets

$$\begin{aligned} &c^2 + 2cba + 2/3ca^3 - b^3 - 1/2b^2a^2 - 1/12ba^4 - 1/72a^6,\\ &yb + 1/2ya^2 - c + 1/6a^3,\\ &yc - 1/6ya^3 + 2ca - b^2 - 1/12a^4, &(16)\\ &y^2 + 2ya - b + 1/2a^2,\\ &z,\\ &x - y - a,\end{aligned}$$

which is quite similar to the case $x = 0$. One first checks whether the first relation between the parameters holds. Then one solves the next three equations for y and then solves the last relation for x. If the system is consistent we have $z = 0$. If x, y turn out to be nonnegative set $u = 1$ and otherwise $u = -1$.

This rules out all cases of vanishing variables. In order to predict when strictly positive solutions exist we are reduced to the cases ($a^2/2 = b$, $a^3/6 \neq c$) and ($a^2/2 \neq b$). Let us consider first the case ($a^2/2 = b$, $a^3/6 \neq c$). Then we have a Groebner basis $b - 1/2a^2, y^3 - 4c + 2/3a^3, z - y/2 + a, x + z - y - a$. It becomes obvious that in order to have a nonnegative solution, we need

$$\begin{aligned} y^3 = 4(c - a^3/6) &\geq 0, \\ z = \left(4(c - a^3/6)\right)^{1/3}/2 + a &\geq 0, \\ x = \left(4(c - a^3/6)\right)^{1/3}/2 &\geq 0, \end{aligned}$$

which simplifies to the two conditions $c - a^3/6 \geq 0, \left(4(c - a^3/6)\right)^{1/3}/2 + a \geq 0$. These are conditions that can easily be checked for given a, b, c and determine existence of a nonnegative solution (x, y, z) of the system (1). Now let us move to the most general situation $b - a^2/2 \neq 0$. In particular, $y \neq 0$ then. Theorem 3 asserts that the Sturm sequence $\{p_i(y)\}$ corresponding to

$$\begin{gathered} f(y) = y^4 + 4y^2(b - a^2/2) + 4y(ba - c - 1/3a^3) - \\ b^2 + ba^2 - a^4/4 = 0 \end{gathered}$$

counts the zeros of this quartic. In particular, there will be positive solutions for just y if and only if $v(0) - v(\infty) > 0$ since 0 is not a root of the quartic. Now $z = \frac{y^2/2 + ya - 1/2b + 1/4a^2}{y}$. This means, that for positive y, z is positive as long as $y^2/2 + ya - 1/2b + 1/4a^2 > 0$. This parabola has roots in $r_{1,2} = a \pm \sqrt{b + a^2/2}$ where $r_1 \leq r_2$. Since the parabola has positive leading coefficient, z is positive for $y \notin [r_1, r_2]$ if $b + a^2/2 > 0$ and $z > 0$ for all $y > 0$ if $b + a^2/2 < 0$. Similarly, $x = y + a - z = \frac{y^2/2 + 1/2b - 1/4a^2}{y}$. Let $r'_{1,2} = \pm\sqrt{1/2a^2 - b}$ with $r'_1 \leq r'_2$. Hence $x > 0$ if and only if $0 < y \notin [r'_1, r'_2]$ if $a^2/2 > b$ and $x > 0$ for all $y > 0$ if $a^2/2 < b$.

We conclude that in order to have x, y, z all positive at the same time we need to satify the following conditions all at the same time.

$$\begin{aligned} & y^4 + 4y^2(b - a^2/2) + y(-4c + 4ba - 4/3a^3) - \\ & b^2 + ba^2 - a^4/4 = 0, \\ & y \notin [r_1, r_2] \text{ or } r_i \notin \mathbf{R}, \\ & y \notin [r'_1, r'_2] \text{ or } r'_i \notin \mathbf{R}, \\ & y > 0. \end{aligned}$$

4.3 Switching Algorithm

These results pave the way for the following algorithm. The algorithm has as input the current state of a system and as output the recommended value for u, either 1 or -1. The origin is then approached by iterated repetition of the algorithm.

Algorithm 4 *(Dynamical steering of the system to the origin.) Suppose our system is in the state (a, b, c).*

> **Case 1,** $2b = a^2, 6c = a^3$**.** *(Check whether $y = 0$.)*
>
> *If $a \geq 0$, set $u = 1$ for a seconds, at which point the system will have reached the origin. If $a < 0$, let $u = -1$ for a seconds.*
>
> **Case 2.** *(Check whether $x = 0$.)*
>
> *Test the consistency of the system (15) and if consistent solve it. If $y, z \geq 0$ set $u = 1$, otherwise set $u = -1$. If the system (15) is not consistent, set $u = -1$.*
>
> **Case 3.** *(Check whether $z = 0$.)*
>
> *Test the consistency of the system (16) and if consistent solve it. If $x, y \geq 0$ set $u = 1$, otherwise set $u = -1$. If the system (16) is not consistent, set $u = -1$.*
>
> **Case 4.** *($2b = a^2, 6c \neq a^3$, x, y, z all $\neq 0$.)*
>
> *If $6c - a^3 > 0$ and $6c > -11a^3$, let $u = 1$. Else, let $u = -1$.*
>
> **Case 5.** *($2b \neq a^2$, x, y, z all $\neq 0$.)*
> *Let $r_1 = a - \sqrt{b + a^2/2}, r_2 = a + \sqrt{b + a^2/2}$, $r'_2 = \sqrt{a^2/2 - b}$. Let $f(y) = y^4 + 4y^2(b - a^2/2) + y(-4c + 4ba - 4/3a^3) - b^2 + ba^2 - a^4/4$ and compute the corresponding Sturm sequence $\{p_i(y)\}$. Let $I = (0, r_1) \cup (r_2, \infty)$ if $r_i \in \mathbf{R}$ and $(0, \infty)$ else. Let $I' = (r'_2, \infty)$ if $r'_2 \in \mathbf{R}$ and $(0, \infty)$ else. Let $S = I \cap I'$.*
>
> *Using the Sturm sequence compute the number of solutions of $f(y)$ in S. If this number is positive, set $u = 1$ and otherwise set $u = -1$.*

References

[1] T. Becker and V. Weispfenning, *Groebner Bases*, Springer-Verlag, New York, 1993.

[2] Michel Coste, *Real Algebraic Geometry*, Notes for the MSRI Workshop, 1998.

[3] D. Cox, J. LIttle, and D. O'Shea, *Ideals, Varieties, and Algorithms* (Second Edition), Springer-Verlag, New York, 1997.

[4] E.B. Lee and L. Markus, *Foundations of Optimal Control Theory*, Malabar, FL:Krieger, 1967.

[5] D. Grayson and M. Stillman, *Macaulay 2*, available at www.math.uiuc.edu/Macaulay2/.

[6] L.Y Pao and G.F. Franklin, "Proximate time-optimal control of third-order servomechanisms," *IEEE Transactions on Automatic Control* **38** (1993), 560-280.